p-adic superspaces and Frobenius.

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Abstract

The notion of a p-adic superspace is introduced and used to give a transparent construction of the Frobenius map on p-adic cohomology of a smooth projective variety over $\mathbb{Z}_p$ (the ring of p-adic integers), as well as an alternative construction of the crystalline cohomology of a smooth projective variety over $\mathbb{F}_p$ (finite field with p elements).

1 Introduction.

If X is a smooth projective variety over $\mathbb{Z}$ or, more generally, over the ring of p-adic integers $\mathbb{Z}_p$ one can define the Frobenius map on the de Rham cohomology of X with coefficients in $\mathbb{Z}_p$ [1]. This map plays an important role in arithmetic geometry (in particular it was used in the Wiles' proof of Fermat's Last Theorem); recently it was used to obtain interesting results in physics [8, 16]. However, the construction of this map is not simple, the usual most invariant approach is based on the consideration of the crystalline site [1]. In any variation one uses the notion of a DP-ideal, that is an ideal I in a ring A with the key property that for $x \in I$, $x^n/n!$ makes sense. To be precise, one assumes the existence of operations $\gamma_n : I \to A$, for $n \geq 0$, that mimic the operations $x \mapsto x^n/n!$ and satisfy the same conditions (for instance $n!\gamma_n(x) = x^n$). The ring A is then called a DP-ring (DP stands for divided powers), and a DP-morphism is a ring homomorphism compatible with the DP-structure.

The advantage of the crystalline cohomology ([5] is a good review) of a scheme X over $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$ is that the coefficients of the theory are in $\mathbb{Z}_p$, though the original X was defined over $\mathbb{F}_p$. Furthermore the action of the Frobenius endomorphism exists in this theory. DP neighborhoods play an essential role essential in defining crystalline cohomology; roughly speaking a DP neighborhood $\widetilde{X}$ of Y in X is described locally by a pair $(\hat{B}, \hat{I})$ where

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$\hat{B}$ is the ring of functions on $\widetilde{X}$ and $\hat{I}$ is the ideal of the subvariety X, the important requirement is that $\hat{I}$ is in fact a DP ideal. However, to construct crystalline cohomology and to analyze its relation to de Rham cohomology some technical problems must be overcome. We will show that using the ideas of supergeometry[1] we can make the exposition less technical, but still completely rigorous. (Idea: Grassmann rings have divided powers naturally and at the same time there are enough of them to "feel" the entire DP neighborhood.) Thus in an appropriate (Grassmann) setting the standard notion of infinitesimal neighborhood replaces the DP envelope.

To summarize, the goals of this paper are as follows. Use supergeometry to give an alternative definition of crystalline cohomology of a smooth projective variety over $\mathbb{F}_p$. And secondly, to define a lifting of the action of Frobenius to the usual de Rham cohomology of a smooth projective variety over $\mathbb{Z}_p$. Our construction of the Grassmann neighborhood may also be applied in the case when X is not smooth over $\mathbb{F}_p$, but in this case it is not likely that our cohomology coincides with the crystalline one. However, the crystalline cohomology is known not to give good answers in the non-smooth (over $\mathbb{F}_p$) case anyway, see [12] for a better cohomology. By playing with our definitions it may be possible to create a theory that gives good results in the non-smooth case.

Our considerations are based on the notion of a p-adic superspace defined as a covariant functor on an appropriate subcategory of the category of $\mathbb{Z}_p$-algebras. It seems that this notion is interesting in itself; one can hope that it can be used to introduce and analyze "p-adic supersymmetry" and "p-adic superstring" making contact with the p-adic B-model of [8] and p-adic string theory (see [3] for a review).

The notion of a p-adic superspace that we use is very general; it is impossible to obtain any significant results in such generality. In all of our examples however, we consider functors defining a superspace that are prorepresentable in some sense. We sketch the proof of this fact in the Appendix and show how to use it to get a more conceptual derivation of some of our statements.

Let us note that a lot of what follows does not work for the even prime and so we omit that case by default. Some modifications designed to allow for the even prime are mentioned in Sec. 4.2.

Finally, any functor, by default, is a covariant functor. All the rings are assumed to be unital. All varieties are of finite type over $\mathbb{Z}_p$.

[1] A reader who is unfamiliar with supermathematics is encouraged to glance through Sec. 1.2.

1.1 Summary of main definitions and results.

The category Λ that serves as the source for most of our functors is described in Definition 2.1. The main notion in this paper is that of a p-adic superspace (see Definition 3.1). The replacement for a DP neighborhood in $\mathbb{P}^n_{\mathbb{Z}_p}$ of $W \subset \mathbb{P}^n_{\mathbb{F}_p}$, namely $\widetilde{W}$ (called the Grassmann neighborhood), is given in Definition 3.12. Finally, our notion of de Rham cohomology of a p-adic superspace is summarized in Definition 4.3.

The appendix contains a discussion of prorepresentable p-adic superspaces and should be considered as the correct general setting for this paper.

For the Frobenius action on $\widetilde{W}$ please see Lemma 3.13. The investigation of functions on $\widetilde{pt}$ (where pt is a point in a line or $\mathbb{P}^1$) is undertaken in Sec. 4.1 and summarized in Corollary 6.6.

The key points of the paper are the comparison results between our de Rham cohomology of p-adic superspaces and the usual de Rham cohomology as well as the crystalline cohomology. Namely, for a smooth V over $\mathbb{Z}_p$, Lemma 4.5 relates the de Rham cohomology of the completion of V with respect to p and the de Rham cohomology of the p-adic superspace associated to V. Lemma 4.11 shows that when V is also projective, the completion can be removed. The main theorem of the paper is Theorem 4.15 that establishes the isomorphism between the de Rham cohomology of a smooth projective V over $\mathbb{Z}_p$ and the de Rham cohomology of the Grassmann neighborhood in $\mathbb{P}^n$ of the p-adic superspace associated to V. Corollary 4.17 compares the crystalline cohomology of W, a smooth variety over $\mathbb{F}_p$, with the de Rham cohomology of its Grassmann neighborhood $\widetilde{W}$.

Theorem 4.15 is refined in Theorem 5.5 where it is shown that the isomorphism is in fact compatible with the Hodge filtration on one side and a very natural filtration on the other. This allows us to re-prove some divisibility estimates for the action of Frobenius on the de Rham cohomology of V. The most useful of these is stated in Lemma 5.6.

The main motivation for this paper is Corollary 4.16 that shows that a smooth projective variety over $\mathbb{Z}_p$ has an action of Frobenius on its de Rham cohomology.

1.2 Supermathematics.

The present paper aims to explain crystalline cohomology from the point of view of supergeometry. However the knowledge of supergeometry itself is not essential for the reading of this text (at least formally). Our treatment of spaces as functors allows for a quick jump from the familiar commutative

setting to the supercommutative one.

The functorial approach to the definition of the superspace (in an essentially different form) was advocated in [15] and [7]. It is analogous to the functor approach to the theory of schemes in algebraic geometry. An excellent reference for this point of view is [6] where this theory is developed from the very beginning; groups and Lie algebras are treated as well. This treatment of algebraic geometry generalizes immediately to the setting of supergeometry by replacing commutative rings with supercommutative ones. The reader is invited to consult these papers in the case of necessity. For a more complete understanding of supermathematics we recommend [9], as well as [2] for those looking for a more abstract and conceptual picture.

The very few concepts that we do need are explained below.

Definition 1.1. By a supercommutative ring A we mean a $\mathbb{Z}/2\mathbb{Z}$-graded ring (i.e. $A = A^0 \oplus A^1$ with multiplication respecting the grading) such that

$$a_i \cdot a_j = (-1)^{ij} a_j \cdot a_i$$

where $a_i \in A^i$.[2] We say that $a \in A^0$ is *even* and $b \in A^1$ is *odd*.

Remark. Any commutative ring A is also supercommutative with $A^1 = 0$. We often use freely generated rings, i.e. $B[x_i, \xi_j]$ with B commutative, and generators x_i commuting ($x_i x_j = x_j x_i$) and ξ_j anti-commuting ($\xi_i \xi_j = -\xi_j \xi_i$), that is x_i are even and ξ_i are odd, thus $x_i \xi_j = \xi_j x_i$. Note that specifying the parity of B, x_i and ξ_i is sufficient to define the $\mathbb{Z}/2\mathbb{Z}$-grading. It is convenient to consider x_i and ξ_j as commuting and anticommuting variables.

Definition 1.2. Denote by *Super* the category of supercommutative rings, with morphisms being algebra homomorphisms respecting the grading (i.e. a morphism preserves the $\mathbb{Z}/2\mathbb{Z}$ decomposition). In other words, morphisms are parity preserving homomorphisms.

Remark. To give the reader a feeling for the morphisms in *Super* we point out that $\mathrm{Hom}_{Super}(\mathbb{Z}[x_1, ..., x_n], A) = (A^0)^n$, $\mathrm{Hom}_{Super}(\mathbb{Z}[\xi_1, ..., \xi_n], A) = (A^1)^n$ and $\mathrm{Hom}_{Super}(\mathbb{Z}[x_1, ..., x_n, \xi_1, ..., \xi_m], A) = (A^0)^n \times (A^1)^m$. We note the standard use of x_i for even variables and ξ_i for odd.

The advantage of the functorial approach to spaces is that it is deceptively simple. We can say that a superspace, in the most general sense, is a

[2] This is the sign rule. Whenever two odd objects are to be written in the opposite order, a minus sign is the price. This is the mildest possible modification of the usual commutativity.

functor from *Super* (or from a subcategory of *Super*) to *Sets* (the category of sets). Morphisms are natural transformations of functors. This definition ignores the most important aspect of spaces, namely that they should be local. This can be included in the definition in many ways. For the general method see [6] and compare with our approach based on the consideration of the body of a superspace (Definition 3.2) and prorepresentability (Appendix). Supergroups are functors from *Super* to *Groups* (the category of groups), etc. The action of a supergroup G on a superspace X can be defined very naturally: for a supercommutative ring A we should have an action of the group $G(A)$ on the set $X(A)$ satisfying some conditions of functoriality.

We define the rest of the concepts as needed.

In fact the reader is already quite familiar with some of the supermathematics that we use. For example, the de Rham complex of a smooth affine variety V is a supercommutative algebra with a differential. This algebra can be interpreted as the algebra of functions on the odd tangent space ΠTV of V and the differential as an action of the odd line $\mathbb{A}^{0,1}$ on ΠTV. This is thoroughly discussed in Sec. 4.

1.3 Main constructions.

Let us explain our constructions in some detail. (The reader can skip this explanation. However it could be useful for some people, in particular, for readers with a background in mathematical physics.)

If a (projective or affine) variety is specified by means of equations with coefficients belonging to a commutative ring R it makes sense to consider the unknowns as belonging to any R-algebra. This means that every variety of this kind (variety over R) specifies a functor on the category of R-algebras with values in the category of sets.

This remark prompts a preliminary definition of a superspace over a ring R as a functor taking values in the category of sets and defined on the category of $\mathbb{Z}/2\mathbb{Z}$-graded supercommutative R-algebras or better yet on its subcategory Λ. (Morphisms are parity preserving homomorphisms.) We define a map of superspaces as a map of functors; this definition depends on the choice of the category Λ. In particular, in the case when R is a ring of p-adic integers $\mathbb{Z}_p$ we take as Λ the category of supercommutative rings of the form $B \otimes \Lambda_n$, where B is a finitely generated commutative ring in which the multiplication by p is nilpotent and B/pB does not contain nilpotent elements. Here Λ_n stands for the Grassmann ring (free supercommutative ring with n odd generators.) Every ring in the category Λ can be considered as a $\mathbb{Z}_p$-algebra since multiplication by a series $\sum_n a_n p^n$ is well defined

because multiplication by p is nilpotent. Considering functors on the above category we come to the definition of a p-adic superspace.

The typical example of a superspace is $\mathbb{A}^{n|k,m}$ corresponding to a functor, that sends an R-algebra to a set of rows with n even, k even nilpotent elements and m odd elements of the algebra. A function on a superspace is a map of this superspace to $\mathbb{A}^{1,1} = \mathbb{A}^{1|0,1}$. In the p-adic case the functions on $\mathbb{A}^{n|k,m}$ are described in Theorem 3.8. In particular, the functions on $\mathbb{A}^{0|1,0}$ correspond to series with divided powers $\sum a_n \frac{x^n}{n!}$ where $a_n \in \mathbb{Z}_p$. (Notice that in this statement it is important that the functors are defined on the category Λ described above. We could consider the functors on the larger category of all supercommutative $\mathbb{Z}_p$-algebras; then only a series of the form $\sum a_n x^n$ with $a_n \in \mathbb{Z}_p$ corresponds to a function.)

The definition of a superspace in terms of a functor is too general; one should impose some additional restrictions to develop an interesting theory. One of the possible ways is to impose some conditions on the "bosonic part" of the superspace (i.e., on the restriction of the functor to commutative R-algebras.) In particular, we notice that the category Λ we used in the p-adic case contains the category of commutative $\mathbb{F}_p$-algebras without nilpotent elements; in this case the restriction of the functor specifying a p-adic superspace Y should correspond to a variety over $\mathbb{F}_p$ (the body $[Y]$ of the superspace Y).

For any $C \in \Lambda$, let us set $\rho(C) = C/C^{nilp}$, where C^{nilp} is the ideal of nilpotent elements in C. In our case C^{nilp} is generated by p and the odd generators. From definitions we see that $\rho(C)$ is an $\mathbb{F}_p$-algebra. Thus the natural projection $C \to \rho(C)$ induces a map $\pi : Y(C) \to [Y](\rho(C))$. For every (open or closed) subvariety $Z \subset [Y]$ we can define a p-adic subsuperspace $Y|_Z \subset Y$ as a maximal subsuperspace of Y having Z as its body. More explicitly, $Y|_Z(C) = \pi^{-1}(Z(\rho(C)))$; we can say that $Y|_Z$ is a subsuperspace of Y over Z.

We give a definition of the body only in p-adic case, but similar constructions also work in other situations.

For every superspace Y over a ring R we can introduce a notion of a differential form and of the de Rham differential. (Differential forms are defined as functions on the superspace ΠTY that parameterizes the maps from the superspace $\mathbb{A}^{0,1} = \mathbb{A}^{0|0,1}$ to Y.) One can try to define the cohomology of Y as cohomology of the differential R-module $\Omega(Y)$ of differential forms on Y, but this definition does not capture the whole picture (it corresponds to the consideration of the Hodge cohomology $H^{k,0}$). The right definition of de Rham cohomology of Y can be given in terms of hypercohomology of a sheaf of differential R-modules on the body of Y. (To an open subset $Z \subset [Y]$ we

assign the module $\Omega(Y|_Z)$ of differential forms on $Y|_Z$.)

Now we are able to define our analogue of the crystalline cohomology of a projective $\mathbb{F}_p$-variety $X \subset \mathbb{P}^n_{\mathbb{F}_p}$ where $\mathbb{P}^n_{\mathbb{F}_p}$ stands for projective space over $\mathbb{F}_p$. One can regard $\mathbb{P}^n_{\mathbb{F}_p}$ as a body of the p-adic projective superspace $\mathbb{P}^n$; this remark permits us to consider $\mathbb{P}^n|_X$ (the subsuperspace of $\mathbb{P}^n$ over X). We define "crystalline" cohomology of X as the de Rham cohomology of the p-adic superspace $\mathbb{P}^n|_X$. The Frobenius map Fr acts naturally on this cohomology. (The usual action of Fr on $\mathbb{P}^n$ sending every homogeneous coordinate to its p-th power preserves the $\mathbb{F}_p$-variety X and therefore $\mathbb{P}^n|_X$.) We prove that for a smooth $\mathbb{F}_p$-variety X the cohomology of $\mathbb{P}^n|_X$ coincides with the p-adic de Rham cohomology of any variety X' over $\mathbb{Z}_p$ that gives X after reduction to $\mathbb{F}_p$ (and therefore the Frobenius acts on the cohomology of X'); see Corollary 4.16.

2 Category Λ.

Consider the local ring $\mathbb{Z}_p$ with the maximal ideal $p\mathbb{Z}_p$ (it is a DP ideal since $p^n/n!$ which is obviously in $\mathbb{Q}_p$ is actually in $\mathbb{Z}_p$ because $\mathrm{ord}_p n! \leq n$). Our main object is the category Λ.

Definition 2.1. Let Λ be the category with objects Λ_B that are super commutative rings freely and finitely generated over a commutative ring B[3] (B is allowed to vary) by odd generators. More precisely, we require that B is a finitely generated commutative ring such that p is nilpotent, and B/pB has no nilpotent elements. We write the $\mathbb{Z}/2\mathbb{Z}$ grading of Λ_B as follows $\Lambda_B = \Lambda_B^0 \oplus \Lambda_B^1$ where Λ_B^0 is even and Λ_B^1 is odd. The morphisms are parity preserving homomorphisms.

The requirement that p be nilpotent is necessary to allow for evaluation of infinite series such as $\sum a_n x^n$ at elements pb with $b \in B$. Also, for example, we see that Λ_B is a $\mathbb{Z}_p$-module (in fact it is a $\mathbb{Z}/p^N\mathbb{Z}$-module for an N large enough). This means that we can consider Λ as a subcategory of the category of $\mathbb{Z}_p$-algebras.

Remark. We use the notation Λ_B to denote a generic element of Λ to emphasize the fact that it is B that is important, not the number of odd variables. The notation is meant to remind the reader of the exterior algebra (Grassmann algebra) where the field has been replaced with B. A more precise statement would be that $\Lambda_B \simeq B[\xi_i]$ where $B[\xi_i]$ is a polynomial ring in

[3] A typical example of such B is $\mathbb{Z}/p^n\mathbb{Z}$.

ξ_is with coefficients in B, however in this case the variables ξ_i are anti-commuting, that is $\xi_i\xi_j = -\xi_j\xi_i$. The parity is determined by the total number of variables ξ_i.

Denote by Λ_B^+ the ideal in Λ_B generated by $\xi_1, ..., \xi_n$. Notice that Λ contains the category of $\mathbb{F}_p$-algebras without nilpotent elements as a full subcategory, and there is a retraction onto it that sends Λ_B to $\Lambda_B/(pB + \Lambda_B^+) = B/pB$. The ideal $pB + \Lambda_B^+$ is to play a very important role for us. One should mention that it can be characterized by the fact that it consists exactly of the nilpotent elements of Λ_B (this follows from the lack of nilpotent elements in B/pB). As a consequence we see that if $A \to \Lambda_B$ is any morphism and $I \subset A$ is a nilpotent ideal, then I is carried to $pB + \Lambda_B^+$ by the morphism. However its most important property is explained in the following Theorem.

Theorem 2.2. *For every $\Lambda_B \in \Lambda$, $pB + \Lambda_B^+ \subset \Lambda_B$ is naturally a DP ideal, i.e. there are operations $\gamma_n : pB + \Lambda_B^+ \to \Lambda_B$ that satisfy the axioms in [1]. Furthermore, any morphism in Λ preserves this structure automatically.*

Remark. Thus every object in Λ is in fact a DP pair $(\Lambda_B, pB + \Lambda_B^+)$ and any morphism preserves the DP structure. This explains our choice of Λ.

Proof. Represent B as $\mathbb{Z}_p[x_i]/I$, where $\mathbb{Z}_p[x_i]$ is the polynomial algebra over $\mathbb{Z}_p$. Note that since $\mathbb{Z}_p$ is torsion free $\Lambda_{\mathbb{Z}_p[x_i]} \subset \Lambda_{\mathbb{Q}_p[x_i]}$ and we can define $\gamma_n : \Lambda_{\mathbb{Z}_p[x_i]} \to \Lambda_{\mathbb{Q}_p[x_i]}$ by $\gamma_n(x) = x^n/n!$ for all n. We claim that $p\mathbb{Z}_p[x_i] + \Lambda^+_{\mathbb{Z}_p[x_i]}$ maps under γ_n to $\Lambda_{\mathbb{Z}_p[x_i]}$. (Thus γ_n define a DP structure on the pair $(\Lambda_{\mathbb{Z}_p[x_i]}, p\mathbb{Z}_p[x_i] + \Lambda^+_{\mathbb{Z}_p[x_i]})$.) To show this it is sufficient to check that $x^n/n!$ is in $\Lambda_{\mathbb{Z}_p[x_i]}$ for $x \in p\mathbb{Z}_p[x_i]$ and for $x \in \Lambda^+_{\mathbb{Z}_p[x_i]}$.

For $x \in p\mathbb{Z}_p[x_i]$, this follows from the observation that $p^n/n! \in \mathbb{Z}_p$. Now suppose that $e \in \Lambda^{0+}_{\mathbb{Z}_p[x_i]}$, let $e = e_1 + ... + e_k$ where e_i are even and of the form $f_i\xi_{i_1}...\xi_{i_s}$, i.e. write e as the sum of monomials in ξ_is. Notice that $e_i^n = 0$ if $n > 1$. So that

$$e^n = (e_1 + ... + e_k)^n = \sum_{\sum n_i = n} \frac{n!}{n_1!...n_k!} e_1^{n_1}...e_k^{n_k} = \sum_{\substack{\sum n_i = n \\ n_i = 0 \text{ or } 1}} n! e_1^{n_1}...e_k^{n_k}.$$

Consider an element $x = e + o \in \Lambda^+_{\mathbb{Z}_p[x_i]}$ with e even and o odd. Then $x^n = (e + o)^n = e^n + ne^{n-1}o$ and we are done.

Since γ_n satisfy the axioms for a DP structure we obtain in this way a DP structure on the pair $(\Lambda_{\mathbb{Z}_p[x_i]}, p\mathbb{Z}_p[x_i] + \Lambda^+_{\mathbb{Z}_p[x_i]})$. Note that $\Lambda_B = \Lambda_{\mathbb{Z}_p[x_i]}/I[\xi_j]$,

thus it inherits a DP structure from $\Lambda_{\mathbb{Z}_p[x_i]}$ if (and only if) $I[\xi_j] \cap (p\mathbb{Z}_p[x_i] + \Lambda^+_{\mathbb{Z}_p[x_i]})$ is preserved by γ_n. But $I[\xi_j] \cap (p\mathbb{Z}_p[x_i] + \Lambda^+_{\mathbb{Z}_p[x_i]}) = I \cap p\mathbb{Z}_p[x_i] + \Lambda^+_I$, and clearly Λ^+_I is preserved by γ_n. As for $I \cap p\mathbb{Z}_p[x_i]$, if $pf \in I$ then $(pf)^n/n! = pfg$ with $g \in \mathbb{Z}_p[x_i]$ thus $(pf)^n/n! \in I$. We conclude that Λ_B inherits a DP structure on $pB + \Lambda^+_B$.

Next we observe that the DP structure obtained as above does not depend on a particular representation of B as a quotient of a polynomial algebra. Namely, if we represent B as $\mathbb{Z}_p[y_j]/J$ and obtain a DP structure on Λ_B in that way, then the identity map on Λ_B lifts to a homomorphism from $\Lambda_{\mathbb{Z}_p[x_i]}$ to $\Lambda_{\mathbb{Z}_p[y_j]}$ (because $\Lambda_{\mathbb{Z}_p[x_i]}$ is free) that is automatically compatible with DP structure (since γ_n is just $x^n/n!$). Because the projections to Λ_B are DP compatible by definition, the identity map is DP compatible as well.

If $\Lambda_B \to \Lambda_C$ is any morphism then it lifts to a morphism of the free algebras that cover Λ_B and Λ_C as above. The lifting is again automatically compatible with DP structure, ensuring that $\Lambda_B \to \Lambda_C$ is a DP morphism. $\square$

Because of the nature of our definition of DP structure on Λ_B we use the more suggestive $x^n/n!$ instead of the more accurate γ_n to denote the DP operations. As we have shown above the symbol $x^n/n!$ is functorially defined for elements of the ideals $pB + \Lambda^+_B$.

Remark. Note that a sufficient condition on J for $\Lambda_{\mathbb{Z}_p[x_i]}/J$ to inherit a DP structure is that it be a ξ-homogeneous ideal, i.e. $J = \bigoplus_\alpha J \cap \mathbb{Z}_p[x_i]\xi^\alpha$, where α is a multi-index. An example used in the Theorem above is $J = I[\xi_j]$. Perhaps one can use this observation to enlarge the category Λ.

Remark. Given any super-commutative $\mathbb{Z}_p$-algebra $A = A^0 \oplus A^1$, we may define $A^+ \subset A$ by $A^+ = pA + A^1 A$, generalizing the ideal $pB + \Lambda^+_B \subset \Lambda_B$. This ideal is functorial, however it is not clear why it should have any DP structure. Various conditions may be imposed to ensure this. The previous remark is an example.

3 Superspaces, neighborhoods and Frobenius.

We would like to base our definition of a p-adic superspace on the notion of a functor from Λ to *Sets*, the category of sets. Since we are interested in studying geometric objects, we would like to impose conditions that would make the functor "local", the easiest way to do it is through the notion of the body of a p-adic superspace.

Definition 3.1. A p-adic superspace X is a functor (covariant) from Λ to $Sets$, such that the restriction to the full subcategory of $\mathbb{F}_p$-algebras without nilpotent elements corresponds to a variety $[X]$ over $\mathbb{F}_p$.

Definition 3.2. The body of a p-adic superspace X is the variety $[X]$.

Definition 3.3. A map $\alpha : X \to Y$ of superspaces is a natural transformation from X to Y.

A more familiar object, the purely even superspace, is obtained by requiring that the functor factors through Λ^0, the category with objects of the form Λ_B^0.

We have the usual functors $\mathbb{A}^n$ and $\mathbb{P}^n$,[4] where

$$\mathbb{A}^n(\Lambda_B) = \{(r_1, ..., r_n) | r_i \in \Lambda_B^0\}$$

and

$$\mathbb{P}^n(\Lambda_B) = \{(r_0, ..., r_n) | r_i \in \Lambda_B^0, \sum \Lambda_B r_i = \Lambda_B\}/(\Lambda_B^0)^\times.$$

Note that these are purely even. More generally we can define the superspace

$$\mathbb{A}^{n,m}(\Lambda_B) := \{(r_1, ..., r_n, s_1, ..., s_m) | r_i \in \Lambda_B^0, s_i \in \Lambda_B^1\}.$$

A further generalization that we will need is

$$\mathbb{A}^{n|k,m}(\Lambda_B) := \{(r_1, ..., r_n, t_1, ..., t_k, s_1, ..., s_m) | r_i \in \Lambda_B^0, t_i \in pB + \Lambda_B^{0+}, s_i \in \Lambda_B^1\},$$

in other words r_i are even elements, t_i are even nilpotent and s_i are odd elements. One can also define

$$\mathbb{P}^{n,m}(\Lambda_B) := \{(r_0, ..., r_n, s_1, ..., s_m) | r_i \in \Lambda_B^0, s_i \in \Lambda_B^1, \sum \Lambda_B r_i = \Lambda_B\}/(\Lambda_B^0)^\times$$

but we will not need it.

Remark. The most important cases from the above are

$$\mathbb{A}^{1|0,0}(\Lambda_B) = \Lambda_B^0$$

$$\mathbb{A}^{0|1,0}(\Lambda_B) = pB + \Lambda_B^{0+}$$

$$\mathbb{A}^{0|0,1}(\Lambda_B) = \Lambda_B^1$$

they are the main building blocks for the theory in this paper.

[4]These functors, and others like them below, can actually be defined as usual superspaces, i.e. they can be obviously extended to the category of all supercommutative rings. Here we use their restriction to Λ, but no extra structure of Λ is required.

Definition 3.4. A function on a p-adic superspace X is a natural transformation from X to the superline $\mathbb{A}^{1,1}$.

Considering all natural transformation from X to the superline $\mathbb{A}^{1,1}$ we obtain the set of functions on X. It is easily seen to be a ring by observing that the functor $\mathbb{A}^{1,1}$ takes values in the category of supercommutative rings.

A very versatile notion that we will need is that of a restriction of a p-adic superspace Y to a subvariety Z (it need not be open or closed) of its body. It is the maximal subsuperspace of Y with body Z. More precisely we have the following definition.

Definition 3.5. Let Y be a p-adic superspace and Z a subvariety of $[Y]$, then the p-adic superspace $Y|_Z$ is defined to make the following diagram cartesian.

$$\begin{tikzcd} Y|_Z(\Lambda_B) \arrow[r, hook] \arrow[d] & Y(\Lambda_B) \arrow[d, "\pi"] \\ Z(B/pB) \arrow[r, hook, "i"] & Y(B/pB) \end{tikzcd}$$

Consider the following "local" analogue of functions on X.

Definition 3.6. Define the pre-sheaf of rings $\mathcal{O}_X$ on $[X]$ by setting $\mathcal{O}_X(U)$ to be the ring of functions on $X|_U$, for any open U in $[X]$.

There is no reason to expect that the pre-sheaf $\mathcal{O}_X$ is a sheaf in general. Thus the usual thinking about functions in terms of coordinates is not advised. However, it is a sheaf for all the superspaces that we consider in this paper. If one wants a more general setting in which $\mathcal{O}_X$ is a sheaf, one should consider prorepresentable superspaces as defined in the Appendix.

Definition 3.7. Denote by $R\langle y_1, ..., y_k\rangle$ the ring whose elements are formal expressions $\sum_{K\geq 0} a_K y^K/K!$, where $a_K \in R$ and y_is are commuting variables.[5] Note that $K!$ need not be invertible in R. We call $R\langle y_1, ..., y_k\rangle$ the ring of power series with divided powers.

Remark. Clearly we can add such expressions, but it is also easy to see that we can multiply them since $\left(\sum a_i y^i/i!\right)\left(\sum b_j y^j/j!\right) = \sum\left(\sum_{i+j=n} \frac{n!}{i!j!} a_i b_j\right) y^n/n!$.

Theorem 3.8. *The functions on $\mathbb{A}^{n|k,m}$ are isomorphic as a ring to*

$$\left\{\sum_{I,J,T\geq 0} a_{I,J,T} x^I \xi^J y^T/T!\right\}$$

[5] Here and below I, J, K, T are multi-indices and $T! = t_1!t_2!....$

where x_i and y_i are even and ξ_i are odd, and $a_{I,J,T} \in \mathbb{Z}_p$ with $a_{I,J,T} \to 0$ as $I \to \infty$.

Proof. Recall that $\mathbb{A}^{n|k,m}(\Lambda_B) = \{(r_1, ..., r_n, t_1, ..., t_k, s_1, ..., s_m)|r_i \in \Lambda_B^0, t_i \in pB + \Lambda_B^{0+}, s_i \in \Lambda_B^1\}$, then every $\sum_{I,J,T\geq 0} a_{I,J,T} x^I \xi^J y^T/T!$ can be evaluated at every (r,t,s) by setting $x = r$, $\xi = s$ and $y = t$ to obtain an element of $\Lambda_B = \mathbb{A}^{1,1}(\Lambda_B)$.

Furthermore, $\mathbb{A}^{n,m}$ is pro-represented[6] by $(\mathbb{Z}_p/p^N\mathbb{Z}_p)[x_1, ..., x_n, \xi_1, ..., \xi_m] \in \Lambda$ (they are in Λ because $\mathbb{F}_p[x_i]$ has no nilpotent elements). Here x_i are even and ξ_i are odd. Thus[7] the functions are

$$\varprojlim_N (\mathbb{Z}_p/p^N\mathbb{Z}_p)[x_1, ..., x_n, \xi_1, ..., \xi_m]$$

i.e. of the form $\sum_{I,J\geq 0} a_{I,J} x^I \xi^J$ with $a_{I,J} \to 0 \in \mathbb{Z}_p$ as $I \to \infty$. The case of $\mathbb{A}^{0|k,0}$ is not as trivial, the issue is that it is "pro-represented" by $(\mathbb{Z}_p/p^N\mathbb{Z}_p)\langle y_1, ..., y_k\rangle$ but these are not in Λ (in $\mathbb{F}_p\langle y_i\rangle$ all y_i are *nilpotent*). Thus the proof of the complete Theorem is postponed until it appears as Corollary 4.13. □

Definition 3.9. Given a variety V over $\mathbb{Z}_p$ (which can be viewed as a functor from the category of commutative $\mathbb{Z}_p$-algebras to *Sets*) we can define the associated p-adic superspace X_V by setting $X_V(\Lambda_B) = V(\Lambda_B^0)$.

Note that information is lost in passing from the variety to its associated superspace. More precisely, V and its p-adic completion $\hat{V}_p$ will have the same associated superspace. (See Lemma 3.10 below; we return to this discussion in Sec. 4.) This is best illustrated by considering the functions on $\mathbb{A}^n$. As a variety over $\mathbb{Z}_p$ its functions are by definition $\mathbb{Z}_p[x_1, ..., x_n]$, however when considered as a p-adic superspace one gets the much larger ring

$$\varprojlim_N (\mathbb{Z}_p/p^N\mathbb{Z}_p)[x_1, ..., x_n]$$

of functions[8]. Observe that the functions on the purely odd affine space $\mathbb{A}^{0,m}$ do not change. The crucial point for us is the metamorphosis that the functions on $\mathbb{A}^{0|1,0}$ undergo, as we pass from considering it as a variety over $\mathbb{Z}_p$

[6]The phrase F is pro-represented by C_n is used here in a more narrow sense than in the Appendix. Namely, we mean that $F = \varinjlim h_{C_n}$, where $h_{C_n} = \mathrm{Hom}(C_n, -)$, and C_n form an inverse system of objects in the category. A good reference on pro-representable functors (in this sense) in (non-super) geometry is [13].

[7]This is an application of the Yoneda Lemma.

[8]It consists of series with p-adically vanishing coefficients.

(prorepresented by $\mathbb{Z}_p[[x]]$) to the associated superspace; they transform from power series to divided power series. It is this observation that motivates the present paper.

Remark. Our use of $\mathbb{A}^{n,m}$ for a p-adic superspace is somewhat misleading as that symbol is standard for a superscheme; in particular $\mathbb{A}^n$ usually denotes (with the ground ring being implicitly $\mathbb{Z}_p$) $\mathrm{Spec}(\mathbb{Z}_p[x_1, ..., x_n])$. There is no confusion however if we make explicit wether we are discussing a variety or a p-adic superspace associated to it. When we need to make the distinction explicit, we use X_V for the p-adic superspace associated to a variety V.

Lemma 3.10. *Let V be a variety over $\mathbb{Z}_p$, then the body of X_V is the restriction of V to $\mathbb{F}_p$, i.e.,*

$$[X_V] = V|_{\mathbb{F}_p}$$

and the functions on X_V (as a sheaf on $[X_V]$) are given by the completion of the functions on V at the subvariety $V|_{\mathbb{F}_p}$, i.e.,

$$\mathcal{O}_{X_V} = \widehat{(\mathcal{O}_V)}_p$$

thus making precise the difference between V and X_V.

Proof. That $[X_V] = V|_{\mathbb{F}_p}$ is immediate from the definition. The question of functions is local, so assume $V = \mathrm{Spec} A$. Then X_V is pro-represented by $\varprojlim A/p^n A$, so that $\mathcal{O}_{X_V} = \widehat{A}_p$. □

One of the most important notions of this paper is that of the infinitesimal neighborhood of one p-adic superspace inside another. It is meant to replace the DP-neighborhood.

Definition 3.11. Let $X \subset Y$ be p-adic superspaces, define $\widetilde{X}$, the infinitesimal neighborhood of X in Y by $\widetilde{X} = Y|_{[X]}$.[9]

Example. Let $\mathbb{A}^n \hookrightarrow \mathbb{A}^{n+1}$ be an inclusion of p-adic superspaces, i.e. $\mathbb{A}^n(\Lambda_B) = \Lambda_B^0 \times ... \times \Lambda_B^0 \to \Lambda_B^0 \times ... \times \Lambda_B^0 \times \{0\} \subset \mathbb{A}^{n+1}(\Lambda_B)$, then it follows directly that $\mathbb{A}^{n+1}|_{[\mathbb{A}^n]}(\Lambda_B) = \Lambda_B^0 \times ... \times \Lambda_B^0 \times (pB + \Lambda_B^{0+})$, so that $\widetilde{\mathbb{A}^n} = \mathbb{A}^{n|1}$.

Suppose that $W \subset \mathbb{P}^n_{\mathbb{F}_p}$ is a possibly non-smooth $\mathbb{F}_p$-variety. We want to define the notion replacing that of a DP neighborhood of W in $\mathbb{P}^n_{\mathbb{Z}_p}$. We do this as follows. The inclusion of varieties over $\mathbb{F}_p$ gives rise to a subvariety W of the body of the p-adic superspace $\mathbb{P}^n$. Let us use the same notation as above, namely $\widetilde{W}$ to denote $\mathbb{P}^n|_W$, this is the infinitesimal neighborhood of W that behaves much better than W itself.

[9]Note that $\widetilde{X}$ depends only on Y and $[X]$, compare with Definition 3.12.

Definition 3.12. We call $\widetilde{W}$ as above, the DP neighborhood of W in $\mathbb{P}^n$. An alternative name that we sometimes use is Grassmann neighborhood.

Remark. $\widetilde{W}$ is a p-adic superspace whereas W was a variety. While not the same kind of object, we can nevertheless define many notions for p-adic superspaces that we have for varieties. For example as we see in the next section, we may consider the de Rham cohomology of a p-adic superspace.

We have the usual action of the Frobenius map Fr on the p-adic superspace $\mathbb{P}^n$ via raising each homogeneous coordinate to the pth power. The restriction of Fr to the body of $\mathbb{P}^n$ preserves W therefore we have an action of Fr also on $\widetilde{W}$. Summarizing we get:

Lemma 3.13. *Let $W \subset \mathbb{P}^n$ be an inclusion of varieties over $\mathbb{F}_p$, then any lifting of the action of Frobenius from $\mathbb{P}^n_{\mathbb{F}_p}$ to $\mathbb{P}^n_{\mathbb{Z}_p}$ (i.e., a choice of homogeneous coordinates) restricts to an action of Frobenius on $\widetilde{W}$.*

4 De Rham cohomology of superspaces.

Let us briefly review the notion of de Rham cohomology from the point of view of superspaces. This point of view lends itself most naturally to a generalization applicable in our setting. We begin by introducing the notion of the odd tangent space to a p-adic superspace X.

Remark. For a $\Lambda_B \in \Lambda$, we denote by $\Lambda_B[\xi]$ the ring Λ_B with an adjoined extra odd variable ξ. More precisely, given a supercommutative ring R, we can form $R[\xi]$ by considering expressions of the form $a + b\xi$, with multiplication defined by $(a + b\xi)(c + d\xi) = ac + (ad + (-1)^{\bar{c}}bc)\xi$. Where $a, b, c, d \in R$ and $\bar{c}$ is the parity of c.

Definition 4.1. Let X be a p-adic superspace, define a new p-adic superspace ΠTX, the odd tangent space of X, by

$$\Pi TX(\Lambda_B) = X(\Lambda_B[\xi]).$$

Functions on ΠTX will serve as the differential forms on X. We will define the differential d and the grading on differential forms in terms of an action of a supergroup on ΠTX.

Note that there is a natural map $\Lambda_B[\xi] \to \Lambda_B$ that sends ξ to 0. This defines a map of p-adic superspaces

$$\pi : \Pi TX \to X$$

and a corresponding map on the bodies[10]

$$[\pi] : [\Pi TX] \to [X].$$

The superspace ΠTX carries an action[11] of the supergroup $\mathbb{A}^{0,1} \rtimes (\mathbb{A}^1)^\times$ defined as follows. Let $o \in \Lambda_B^1 = \mathbb{A}^{0,1}(\Lambda_B)$ we need to define the corresponding automorphism of $\Pi TX(\Lambda_B) = X(\Lambda_B[\xi])$. We accomplish that by defining a morphism in Λ from $\Lambda_B[\xi]$ to itself via $\Lambda_B \xrightarrow{Id} \Lambda_B$ and $\xi \mapsto \xi + o$, this induces the required automorphism. Similarly we define the automorphism corresponding to $e \in (\Lambda_B^0)^\times = (\mathbb{A}^1)^\times(\Lambda_B)$ by defining a morphism in Λ via $\Lambda_B \xrightarrow{Id} \Lambda_B$ and $\xi \mapsto e\xi$.

At this point, for the sake of concreteness, let us assume that X is prorepresentable (in the sense of Definition 6.2). This is always the case in our setting. We are now ready to define the differential graded sheaf $\Omega_{X/\mathbb{Z}_p}$ of $\mathbb{Z}_p$-modules on $[X]$ the body of X. Its hypercohomology will be called the de Rham cohomology of X. We denote it by $DR_{\mathbb{Z}_p}(X)$.

Let $U \subset [X]$ be an open subvariety, consider the $\mathbb{Z}_p$-algebra of functions on $\Pi T(X|_U)$ (i.e. natural transformations to $\mathbb{A}^{1,1}$). More concisely we have the following.

Definition 4.2. Define the pre-sheaf $\Omega_{X/\mathbb{Z}_p}$ on $[X]$ by setting

$$\Omega_{X/\mathbb{Z}_p} = [\pi]_* \mathcal{O}_{\Pi TX}.$$

Note that $\Omega_{X/\mathbb{Z}_p}$ carries a grading induced by the action of $(\mathbb{A}^1)^\times$. The sections of $\Omega^n_{X/\mathbb{Z}_p}$ should be thought of as differential n-forms on X. We also have a differential[12] d (an operator which comes from the canonical odd element[13] $\eta \in \mathbb{A}^{0,1}(\Lambda_B[\eta])$ for every Λ_B). More precisely, suppose that φ is a function on ΠTX. Recall that this means that for every Λ_B there is a natural map $\varphi : \Pi TX(\Lambda_B) \to \Lambda_B$. Define the function $d\varphi$ by specifying that for every Λ_B, it is the following composition of maps:

$$\Pi TX(\Lambda_B) \xrightarrow{i} \Pi TX(\Lambda_B[\eta]) \xrightarrow{a_\eta} \Pi TX(\Lambda_B[\eta]) \xrightarrow{\varphi} \Lambda_B[\eta] \xrightarrow{c} \Lambda_B$$

where i is induced by the inclusion $\Lambda_B \subset \Lambda_B[\eta]$, a_η is the endomorphism of $\Pi TX(\Lambda_B[\eta]) = X(\Lambda_B[\eta][\xi])$ induced by the endomorphism of $\Lambda_B[\eta][\xi]$ given

[10] When X is purely even, $[\Pi TX] = [X]$.

[11] These definitions become more transparent when one thinks of ΠTX as the superspace parameterizing the maps from $\mathbb{A}^{0,1}$ to X.

[12] Speaking informally, it is induced by the infinitesimal action of $\mathbb{A}^{0,1}$.

[13] In fact η is the canonical element in the odd Lie algebra $\mathbb{A}^{0,1}(\mathbb{Z}_p[\eta])$ of $\mathbb{A}^{0,1}$.

by $\Lambda_B \xrightarrow{Id} \Lambda_B$, $\eta \mapsto \eta$, $\xi \mapsto \xi + \eta$; φ is self explanatory and c reads off the coefficient of η.

One readily checks that d increases the degree by one.[14]

Definition 4.3. Let X be a p-adic superspace. We define the de Rham cohomology of X as the hypercohomology of $\Omega^\bullet_{X/\mathbb{Z}_p}$, i.e.,

$$DR_{\mathbb{Z}_p}(X) = \mathbb{H}([X], \Omega^\bullet_{X/\mathbb{Z}_p}).$$

Note that it is easy to see from the definitions that $DR_{\mathbb{Z}_p}(-)$ is a contravariant functor from the category of superspaces to the category of graded $\mathbb{Z}_p$-modules. Thus any endomorphism of X induces an endomorphism of $DR_{\mathbb{Z}_p}(X)$.

Example. Let us apply the definitions in the simple example of a line. In this case $X(\Lambda_B) = \Lambda_B^0$ and $\Pi TX(\Lambda_B) = X(\Lambda_B[\xi]) = (\Lambda_B[\xi])^0 = \Lambda_B^0 + \Lambda_B^1\xi$. Thus $\Pi TX = \mathbb{A}^{1,1}$ as expected. The body of X is affine, so to compute $DR_{\mathbb{Z}_p}(X)$ we need only compute the cohomology of the complex of $\mathbb{Z}_p$-modules $\Gamma(\mathcal{O}_{\mathbb{A}^{1,1}})$. By Theorem 3.8 we know that as a $\mathbb{Z}_p$-module it is $S \oplus S\xi$ where $S = \{\sum a_i x^i | a_i \in \mathbb{Z}_p, a_i \to 0\}$. The reader is invited to verify that the action of $(\mathbb{A}^1)^\times$ puts S in degree 0 and $S\xi$ in degree 1, while the differential acts by $\xi\partial_x$. Thus $DR_{\mathbb{Z}_p}(X) = H_{dR}(\widehat{\mathbb{A}^1}_p)$.

More generally, by unraveling the definitions we obtain the following two Lemmas that bridge the gap between the p-adic superspace approach and the usual situation.

Lemma 4.4. *Let SpecA be a smooth variety over $\mathbb{Z}_p$, then*

$$DR_{\mathbb{Z}_p}(X_{SpecA}) = H_{dR}(\widehat{SpecA}_p).$$

Proof. Let $X = X_{\mathrm{Spec}A}$, since $[X]$ is affine the left hand side is computed by the complex of global functions on ΠTX.

But $\Pi TX(\Lambda_B) = X(\Lambda_B[\xi]) = Hom_{Super}(A, \Lambda_B[\xi]) = Hom_{Super}(\Omega_A, \Lambda_B)$. Recall that Ω_A is the supercommutative ring generated by a (even) and da (odd) for $a \in A$ subject to the usual relations (most important being the Leibniz Rule).

So ΠTX is prorepresented by Ω_A, thus $\mathcal{O}_{\Pi TX} = \widehat{(\Omega_A)}_p$ (just like in Lemma 3.10). This is exactly the complex that computes the right hand side, but we still need to verify that this identification is compatible with

[14]The connection between these abstract definitions and the usual de Rham complex is made explicit in Lemma 4.4.

the differentials. It is sufficient to check the compatibility with the actions of $\mathbb{A}^{0,1} \rtimes (\mathbb{A}^1)^\times$.

Note that the action of $\mathbb{A}^{0,1}$ on Ω_A is given by the coaction (which is an algebra morphism)

$$\Omega_A \to \Omega_A[\xi]$$

$$a \mapsto a + da\xi, \quad da \mapsto da$$

and the action of $(\mathbb{A}^1)^\times$ by

$$\Omega_A \to \Omega_A[x^{\pm 1}]$$

$$a \mapsto a, \quad da \mapsto dax$$

while

$$Hom_{Super}(\Omega_A, \Lambda_B) = Hom_{Super}(A, \Lambda_B[\xi])$$

$$\{a \mapsto f(a), da \mapsto \varphi(a)\} \leftrightarrow \{a \mapsto f(a) + \varphi(a)\xi\}.$$

It is now straightforward to check that the action on ΠTX is obtained in this case from the one on Ω_A.

□

Lemma 4.5. *Let V be a smooth variety over $\mathbb{Z}_p$. Then*

$$DR_{\mathbb{Z}_p}(X_V) = H_{dR}(\hat{V}_p).$$

Proof. The left hand side is by definition the hypercohomology of $\Omega^\bullet_{X_V/\mathbb{Z}_p}$ on $[X_V]$, while the right hand side is the hypercohomology of $\widehat{(\Omega^\bullet_{V/\mathbb{Z}_p})}_p$ on $V|_{\mathbb{F}_p}$. The two spaces $[X_V]$ and $V|_{\mathbb{F}_p}$ are the same (Lemma 3.10), so the question is about comparing the two complexes of sheaves locally. They are the same by the proof of Lemma 4.4. □

4.1 Functions on $\widetilde{pt}$.

In this section we study the most basic and at the same time the most crucial example of a DP neighborhood, namely that of a point in a line.

Definition 4.6. Denote by $\widetilde{pt}$ the infinitesimal neighborhood of the origin in $\mathbb{A}^1$.

In this section we are concerned with describing explicitly the functions on $\widetilde{pt}$. This is the key step in the subsequent cohomology computations. A more category theory minded reader may wish to visit the Appendix before going any further.

One sees immediately from the definitions that

$$\widetilde{pt}(\Lambda_B) = pB + \Lambda_B^{0+}.$$

This is our old friend $\mathbb{A}^{0|1,0}$, and has a subfunctor that we will denote by $\widetilde{pt}_0$, it is defined by

$$\widetilde{pt}_0(\Lambda_B) = \Lambda_B^{0+}.$$

It is the functions on $\widetilde{pt}_0$, i.e natural transformations to $\mathbb{A}^1$ that we describe first.[15] Let f be one such transformation, our intention is to show that for $w \in \Lambda_B^{0+}$, we have that $f(w) = \sum_{i=0}^{\infty} a_i w^i/i!$ with $a_i \in \mathbb{Z}_p$ determining f.

Remark. It is clear that any expression $\sum_{i=0}^{\infty} a_i w^i/i!$ gives a function as it can be evaluated at any element of $pB + \Lambda_B^{0+}$, i.e. we do have a map from such expressions to functions. However the injectivity and surjectivity of this map remains to be demonstrated below.

First we need a Lemma.

Lemma 4.7. *Let* $w = \xi_{j_1}\xi_{j_2} + ... + \xi_{j_{2k-1}}\xi_{j_{2k}} \in \Lambda^{0+}_{\mathbb{Z}_p/p^N\mathbb{Z}_p}$, *then*

$$f(w) = \sum_{i=0}^{k} a_i w^i/i!$$

and $a_i \in \mathbb{Z}_p/p^N\mathbb{Z}_p$ *depend only on* f.[16]

Proof. We proceed by induction on k. If $k = 0$ then $w = 0$ and so by functoriality of f, $f(w) \in \mathbb{Z}_p/p^N\mathbb{Z}_p$, define a_0 to be $f(w)$ and we are done.

Assume that the Lemma is true for $k \leq n$. Let $k = n+1$,

$$w = \xi_{j_1}\xi_{j_2} + ... + \xi_{j_{2k-1}}\xi_{j_{2k}} =: x_1 + ... + x_k,$$

and setting $f(w) = \sum_I a_I \xi_I$ consider $\xi_I = \xi_{j_1}...\xi_{j_{2i}}$. Note that by functoriality $i \leq k$, and if $i = k$ then there is only one such ξ_I, denote its coefficient by a_i (we have now defined a_{n+1}). If $i < k$ define a map ϕ from $\Lambda_{\mathbb{Z}_p/p^N\mathbb{Z}_p}$ to itself by sending ξ_{j_s} to ξ_s and the rest of ξ's to 0.

[15]Here we do not need $\mathbb{A}^{1,1}$ as everything is purely even.

[16]The a_i are defined inductively in the proof.

If $\xi_I = x_{s_1}...x_{s_i}$ then $\phi(w) = \xi_1\xi_2 + ... + \xi_{2i-1}\xi_{2i}$, so $f(\phi(w)) = ... + a_i\xi_1\xi_2...\xi_{2i-1}\xi_{2i}$ by the induction hypothesis and $\phi(f(w)) = ...+a_I\xi_1\xi_2...\xi_{2i-1}\xi_{2i}$, so that $a_I = a_i$.

If $\xi_I \neq x_{s_1}...x_{s_i}$ then $\phi(w)$ has fewer than i summands yet is of the form $\xi\xi + ... + \xi\xi$ so that we may use the induction hypothesis to conclude that the top degree of $f(\phi(w))$ is less than $2i$ whereas $\phi(\xi_I) = \xi_1\xi_2...\xi_{2i-1}\xi_{2i}$ has degree $2i$, so that $a_I = 0$.

So $f(w) = \sum_I a_I\xi_I = \sum_i a_i x_{s_1}...x_{s_i} = \sum_{i=0}^{n+1} a_i w^i/i!$, and we are almost done. Namely, we demonstrated that any function f, when restricted to $w \in \Lambda^{0+}_{\mathbb{Z}_p/p^N\mathbb{Z}_p}$ can be written as a DP polynomial with coefficients in $\mathbb{Z}_p/p^N\mathbb{Z}_p$ of degree at most k. However, it is immediate that such an expression is unique, since $w^i/i!$ for $i = 0, ..., k$ form a basis of the free $\mathbb{Z}_p/p^N\mathbb{Z}_p$-submodule of $\Lambda^0_{\mathbb{Z}_p/p^N\mathbb{Z}_p}$ that they span.

$\square$

By functoriality of f we obtain, by considering the above Lemma for all N, that the coefficients a_i are given for all N by the images under the natural projection of $a_i \in \mathbb{Z}_p$.

Theorem 4.8. *Let $e \in \Lambda^{0+}_B$, then $f(e) = \sum_{i=0}^{\infty} a_i e^i/i!$, with a_i as above.*

Proof. Let $e \in \Lambda^{0+}_B$, $e = \sum_i b_i e_{i1} e_{i2}$ where e_{ij} are odd monomials in ξ's. Let $p^N B = 0$. Define a map φ from $\Lambda_{\mathbb{Z}_p/p^N\mathbb{Z}_p}$ to Λ_B by $\mathbb{Z}_p/p^N\mathbb{Z}_p \to B$ being the structure morphism, and $\xi_{2i-1} \mapsto b_i e_{i1}$ and $\xi_{2i} \mapsto e_{i2}$, so that $w = \sum_i \xi_{2i-1}\xi_{2i} \mapsto e$. So $f(e) = f(\varphi(w)) = \varphi(f(w)) = \varphi(\sum_{i=0}^{length(w)} a_i w^i/i!) = \sum_{i=0}^{length(e)} a_i \varphi(w)^i/i! = \sum_{i=0}^{length(e)} a_i e^i/i! = \sum_{i=0}^{\infty} a_i e^i/i!$. Here the *length* of an element in Λ^+_B is the minimal number of monomials (in the odd variables) that are needed to write it down. $\square$

Remark. A consequence of this result is that while our choice of w in Lemma 4.7 is fairly arbitrary, for instance one can reorder the coordinates, this does not in any way affect the coefficients a_i.

Now let us consider the functor $\widetilde{pt}(\Lambda_B) = pB + \Lambda^{0+}_B$ itself. We claim that the functions are still of the form $\sum_{i=0}^{\infty} a_i x^i/i!$ with coefficients in $\mathbb{Z}_p$.[17] We reduce to the previous case to prove the following lemma, from which the claim follows easily.

[17] Here in particular we must assume that $p > 2$ otherwise this expression does not define a function in general and the situation becomes more complicated.

Lemma 4.9. *Let $A = \mathbb{Z}_p/p^N\mathbb{Z}_p[x]$ and*

$$w = px + \xi_1\xi_2 + ... + \xi_{2k-1}\xi_{2k} \in pA + \Lambda_A^{0+},$$

then

$$f(w) = \sum_{i=0}^{\infty} a_i w^i/i!$$

and $a_i \in \mathbb{Z}_p$ depend only on f.

Proof. First we need to define $a_i \in \mathbb{Z}_p$. Recall the subfunctor $\widetilde{pt}_0$ of $\widetilde{pt}$ that sends Λ_B to Λ_B^{0+}. If we restrict $\widetilde{pt}_0$ to the subcategory Λ^N of Grassmann rings with coefficients in B, with $p^N B = 0$, then by Theorem 4.8, $f|_{\widetilde{pt}_0}$ determines (and is determined on Λ^N by) $\{a_i^N \in \mathbb{Z}_p/p^N\mathbb{Z}_p\}$. We observe that by functoriality of f we may take the inverse limit over N to obtain $\{a_i \in \mathbb{Z}_p\}$ that determine $f|_{\widetilde{pt}_0}$ on Λ. Let $\widetilde{f}$ be a new function on $\widetilde{pt}$ defined by $\widetilde{f}(e) = \sum a_i e^i/i!$ for $e \in pB + \Lambda_B^{0+}$, so that $\widetilde{f}$ agrees with f on $\widetilde{pt}_0$. We want to show that they agree on w also.

For any N and n, let us define a map ϕ from Λ_A to $\Lambda_{\mathbb{Z}_p/p^N\mathbb{Z}_p}$ by $\xi_i \mapsto \xi_i$ and $x \mapsto \eta_1\eta_2 + ... + \eta_{2n-1}\eta_{2n}$. Note that under this map

$$w \mapsto p(\eta_1\eta_2 + ... + \eta_{2n-1}\eta_{2n}) + \xi_1\xi_2 + ... + \xi_{2k-1}\xi_{2k} \in \Lambda^{0+}_{\mathbb{Z}_p/p^N\mathbb{Z}_p}$$

and so

$$\phi(f(w)) = f(\phi(w)) = \sum a_i(\phi(w))^i/i! = \phi(\widetilde{f}(w)).$$

Note that setting $f(w) = \sum c_i^N x^i$ with $c_i^N \in \mathbb{Z}_p/p^N\mathbb{Z}_p[\xi_j]$, functoriality implies that we have $c_i \in \mathbb{Z}_p[\xi_j]$ such that $f(w) = \sum c_i x^i$ for all N.

To show that $f(w) = \widetilde{f}(w)$ it suffices to consider the following situation. Let $b_i \in \mathbb{Z}_p[\xi_j]$, define $g = \sum b_i x^i \in \Lambda_A$, suppose that

$$0 = \phi(g) = \sum b_i(\eta_1\eta_2 + ... + \eta_{2n-1}\eta_{2n})^i$$

for all n and N. Since if $i \leq n$ then $0 = \phi(g)$ implies that $i!b_i = 0$ we see that $b_i = 0$ in $\mathbb{Z}_p[\xi_j]$ and $g = 0$. Take $g = f(w) - \widetilde{f}(w)$ and we are done. □

Theorem 4.10. *Let $e \in pB + \Lambda_B^{0+}$, then $f(e) = \sum_{i=0}^{\infty} a_i e^i/i!$.*

Proof. Let $e \in pB + \Lambda_B^{0+}$, $e = pb + \sum_i b_i e_{i1}e_{i2}$ where e_{ij} are odd monomials in ξ's and $b \in B$. Suppose that $p^N B = 0$. Define a map φ from Λ_A to Λ_B by $x \mapsto b$, $\xi_{2i-1} \mapsto b_i e_{i1}$ and $\xi_{2i} \mapsto e_{i2}$, so that $w = px + \sum_i \xi_{2i-1}\xi_{2i} \mapsto e$. So that

$$f(e) = f(\varphi(w)) = \varphi(f(w)) = \varphi(\sum a_i w^i/i!) = \sum a_i \varphi(w)^i/i! = \sum a_i e^i/i!.$$

□

Remark. To summarize the above, we have a canonical map from $\mathbb{Z}_p\langle x\rangle$ to functions on $\widetilde{pt}$. This map, as is explicitly shown in the Lemmas above, is surjective. The fact that it is injective follows from the observation at the end of the proof of Lemma 4.7.

4.2 The case of $p = 2$.

As mentioned before the case of the even prime does not fit into the framework described. The issue is that $\sum_{i=0}^{\infty} p^i/i!$ is convergent in $\mathbb{Z}_p$ only for $p > 2$. It follows that for the case $p = 2$, the functions on the Grassmann neighborhood of a point in the line are not simply DP power series with coefficients in $\mathbb{Z}_p$, rather they form a subset of these with some conditions on the coefficients. While it is possible to describe them explicitly one immediately sees that the homotopy of Lemma 4.14 no longer exists. Consequently one can not prove the cohomology invariance of Grassmann thickening.

It seems one can introduce an alternate framework that works for all primes p. We briefly outline it here. The idea is to introduce $\hat{\Lambda}$, an enlargement of our main category Λ which includes Grassmann rings with an infinite number of variables that allow certain infinite sums as elements. More precisely, we consider rings $\Lambda_B = B[\xi_1, \xi_2, ...]$ where elements have the form $\sum b_i w_i$ where $b_i \in B$ and w_i are monomials in ξ's of degree at most N where N is fixed for each element.[18] Thus $\sum \xi_{2i-1}\xi_{2i}$ is an element, while $\sum_{i=1}^{\infty} \prod_{j=1}^{i} \xi_j$ is not.

One does not get the same functions as before for the case of the Grassmann neighborhood of a point in the line[19], but the homotopy of Lemma 4.14 now makes sense and so we can again show the cohomology invariance of Grassmann thickening by modifying all of the arguments accordingly (some of them simplify somewhat).

Finally, note that the very definition of the Grassmann algebra needs modification by the addition of an extra axiom that $\xi^2 = 0$ for ξ odd.

4.3 De Rham cohomology in the smooth case, a comparison.

Recall that to a variety V over $\mathbb{Z}_p$, considered as a functor from $\mathbb{Z}_p$-algebras to sets, we can associate a superspace X_V with $X_V(\Lambda_B) = V(\Lambda_B^0)$. If V is smooth, then we may consider the usual de Rham cohomology of V and

[18] We still require that $p^M B = 0$ for M large, thus the canonical DP ideal is still nilpotent, it need not however be DP nilpotent.

[19] Instead of power series with DP one gets an extra condition that the coefficients tend to 0 in $\mathbb{Z}_p$.

compare it to the $DR_{\mathbb{Z}_p}(X_V)$. In general the two are not the same, however if V is projective then they are isomorphic.

Lemma 4.11. *Let V be a smooth projective variety over $\mathbb{Z}_p$ then*

$$H_{dR}(V) \simeq DR_{\mathbb{Z}_p}(X_V).$$

Proof. By Lemma 4.5 we need only compare $H_{dR}(\hat{V}_p)$ with $H_{dR}(V)$. The fact that they are isomorphic in the projective case was pointed out to us by A. Ogus, and we provide a sketch of a proof. For the relevant facts about formal schemes and the theorem on formal functions we refer to [4].

By definition, the de Rham cohomology $H_{dR}(V)$ is computed as the hypercohomology of the complex $\Omega^\bullet_{V/\mathbb{Z}_p}$, which can be obtained as the cohomology of the total complex associated to the double complex of $\mathbb{Z}_p$-modules $\oplus_I \Omega^\bullet_{V/\mathbb{Z}_p}(U_I)$, where Is are finite subsets $\{i_1, ..., i_s\}$ of $\{0, ..., n\}$, $U_I = U_{i_1} \cap ... \cap U_{i_s}$ and $U_0, ..., U_n$ form an open affine cover of V. The horizontal differentials are de Rham differentials and the vertical ones are Čech differentials.

Clearly we have a morphism of double complexes

$$\alpha : \oplus_I \Omega^\bullet_{V/\mathbb{Z}_p}(U_I) \to \oplus_I \widehat{(\Omega^\bullet_{V/\mathbb{Z}_p}(U_I))}_p$$

and the double complex on the right computes $H_{dR}(\hat{V}_p)$.

The above morphism on the E_1 term becomes

$$\alpha : H^\bullet(\Omega^\bullet_{V/\mathbb{Z}_p}) \to H^\bullet(\widehat{(\Omega^\bullet_{V/\mathbb{Z}_p})}_p)$$

and it factors as follows

$$H^\bullet(\Omega^\bullet_{V/\mathbb{Z}_p}) \to \widehat{H^\bullet(\Omega^\bullet_{V/\mathbb{Z}_p})}_p \to H^\bullet(\widehat{(\Omega^\bullet_{V/\mathbb{Z}_p})}_p).$$

Recall that projective morphisms preserve coherence and so $H^i(\Omega^j_{V/\mathbb{Z}_p})$ is a finitely generated $\mathbb{Z}_p$-module. Because $\mathbb{Z}_p$ is complete, by Theorem 9.7 in [4], for example, we have that the first arrow is an isomorphism. The second arrow is an isomorphism by the theorem on formal functions.

Since α is an isomorphism on E_1, it induces an isomorphism

$$\alpha : H_{dR}(V) \xrightarrow{\sim} H_{dR}(\hat{V}_p).$$

$\square$

Given a smooth projective V over $\mathbb{Z}_p$ we would like to define the action of Fr on its de Rham cohomology. By above it suffices to do so for $DR_{\mathbb{Z}_p}(X_V)$. As explained in Sec. 3, we have an action of Fr on $\widetilde{X_V}$ (the neighborhood of X_V in $\mathbb{P}^n$) and so on $DR_{\mathbb{Z}_p}(\widetilde{X_V})$. Showing that $DR_{\mathbb{Z}_p}(\widetilde{X_V})$ is isomorphic to $DR_{\mathbb{Z}_p}(X_V)$ would accomplish our goal.

Remark. Another consequence of this isomorphism is that the de Rham cohomology of V depends only on $V|_{\mathbb{F}_p}$ because that is true of $\widetilde{X_V}$. This means in particular that for W smooth projective over $\mathbb{F}_p$, the de Rham cohomology of $\widetilde{W}$ (its Grassmann neighborhood in the projective space over $\mathbb{Z}_p$) coincides with the usual crystalline cohomology of W. That is we give a super-geometric construction of the DP envelope of W in $\mathbb{P}^n_{\mathbb{Z}_p}$.

Observe that we have $i : X_V \hookrightarrow \widetilde{X_V}$ thus also a natural map

$$i^* : DR_{\mathbb{Z}_p}(\widetilde{X_V}) \to DR_{\mathbb{Z}_p}(X_V)$$

that we will show is an isomorphism. It suffices to prove that

$$i^* : \Omega_{\widetilde{X_V}/\mathbb{Z}_p} \to \Omega_{X_V/\mathbb{Z}_p}$$

is a quasi-isomorphism of sheaves on $V|_{\mathbb{F}_p}$. Thus the question becomes local and we may assume, after induction on the codimension, that the situation is as follows.

Let $U \subset U'$ be a pair of smooth affine varieties such that U is cut out of U' by a function g on U'. In this case we will show that

$$\widetilde{X}_U = X_U \times \widetilde{pt}$$

i.e. the infinitesimal neighborhood of U in U' is a direct product of the p-adic superspaces X_U (associated to U) and our $\widetilde{pt}$. Compare this with the Example following Definition 3.11 where this is discussed in the case when the function g is linear and U' is an affine space. The general case is demonstrated below. We will assume that $U' = \mathrm{Spec}A$ and $U = \mathrm{Spec}A/g$,

where $g \in A$. Then unraveling the definitions we see that

$$\begin{aligned}
\widetilde{X}_U(\Lambda_B) &= \{\mathrm{Hom}(A, \Lambda_B) | g \mapsto pB + \Lambda_B^+\} \\
&= \bigcup_n \mathrm{Hom}(A/g^n, \Lambda_B) \\
&= \mathrm{Hom}^{cont}(\hat{A}_g, \Lambda_B) \\
&= \mathrm{Hom}^{cont}(A/g[[x]], \Lambda_B) \\
&= \mathrm{Hom}(A/g, \Lambda_B) \times (pB + \Lambda_B^{0+}) \\
&= X_U(\Lambda_B) \times \widetilde{pt}(\Lambda_B)
\end{aligned}$$

where Hom^{cont} denotes continuous homomorphisms ($\hat{A}_g$ is a topological ring and Λ_B is equipped with the discrete topology). The inclusion $X_U \subset \widetilde{X}_U$ is simply $X_U = X_U \times pt \subset X_U \times \widetilde{pt} = \widetilde{X}_U$.

Next we show that the functions on $X_U \times \widetilde{pt}$ are what was expected, namely if $R = \widehat{(A/g)}_p$ then:

Theorem 4.12. *Any natural transformation f from $X_U \times \widetilde{pt}$ to $\mathbb{A}^1$ is given by*

$$f(\phi, e) = \sum_{i=0}^{\infty} \phi(r_i) e^i / i!$$

for any $\phi \in Hom(A/g, \Lambda_B)$ and $e \in pB + \Lambda_B^{0+}$, where $r_i \in R$ depend only on f.

Remark. The proof below remains valid for the case when X is given by $X(\Lambda_B) = \mathrm{Hom}(\mathbb{Z}_p \langle x_i \rangle, \Lambda_B)$ i.e. $R = \mathbb{Z}_p \langle x_i \rangle$. This justifies the induction on the codimension.

Proof. Begin by noting that in the proofs of Lemma 4.9 and Theorem 4.10 we can replace $\mathbb{Z}_p$ by any p-adically complete ring without zero divisors. In particular these results remain valid when $\mathbb{Z}_p$ is replaced by our $R = \widehat{(A/g)}_p$.

Thus let us define a new category $\Lambda(R)$ consisting of Grassmann rings with coefficients in R-algebras with nilpotent p-action. Denote by $\widetilde{pt}_R$ and $\mathbb{A}^1_R$ the restrictions of similarly named functors from Λ to $\Lambda(R)$, so that they are now functors from $\Lambda(R)$ to *Sets*. As before, if f is a natural transformation from $\widetilde{pt}_R$ to $\mathbb{A}^1_R$, then

$$f(e) = \sum_{i=0}^{\infty} r_i e^i / i!,$$

where $e \in pB + \Lambda_B^{0+}$, and $r_i \in R$ depend only on f.

Observe that any natural transformation f from $X_U \times \widetilde{pt}$ to $\mathbb{A}^1$ defines $\widetilde{f} : \widetilde{pt}_R \to \mathbb{A}^1_R$ by $\widetilde{f}(e) = f(\phi, e)$, where $e \in pB + \Lambda_B^{0+} \in \Lambda(R)$ and ϕ is the structure morphism. Of course the R-module structure on B provides us with a $\mathbb{Z}_p$-morphism $\phi : R \to B \to \Lambda_B$, however as ϕ factors through $R/p^N R = (A/g)/p^N(A/g)$ it determines a unique morphism from A/g to Λ_B and so an element of $X_U(\Lambda_B)$. Conversely, any element of $X_U(\Lambda_B)$ that factors though B makes Λ_B into an element of $\Lambda(R)$. By above

$$\widetilde{f}(e) = \sum_{i=0}^{\infty} r_i e^i / i!$$

that is

$$f(\phi, e) = \sum_{i=0}^{\infty} \phi(r_i) e^i / i!$$

for all $\phi : A/g \to \Lambda_B$ that factor through B.

Let (ϕ, e) be an arbitrary element of $X_U \times \widetilde{pt}(\Lambda_B)$, assume that $p^N B = 0$ so that ϕ factors through $R/p^N R$. Let $e = pb + \sum b_i e_{i1} e_{i2}$ and proceed as in the proof of Theorem 4.10. Define a morphism φ from $\Lambda_{(R/p^N R)[x]}$ to Λ_B by

$$\begin{aligned} \phi : R/p^N R &\to \Lambda_B \\ x &\mapsto b \\ \xi_{2i-1} &\mapsto b_i e_{i1} \\ \xi_{2i} &\mapsto e_{i2}. \end{aligned}$$

Consider the element $(\pi, w) \in X_U \times \widetilde{pt}(\Lambda_{(R/p^N R)[x]})$ where

$$\pi : A/g \to \Lambda_{(R/p^N R)[x]}$$

is the projection onto $R/p^N R \subset \Lambda_{(R/p^N R)[x]}$ and

$$w = px + \sum_i \xi_{2i-1} \xi_{2i},$$

then $\varphi(\pi, w) = (\phi, e)$.[20] Thus

$$f(\phi, e) = f\varphi(\pi, w) = \varphi f(\pi, w) = \varphi(\sum \pi(r_i) w^i / i!) = \sum \phi(r_i) e^i / i!.$$

□

[20] One should really write $X_U \times \widetilde{pt}(\varphi)((\pi, w))$, but that is too cumbersome.

We are finally able to complete the proof of Theorem 3.8 which we restate as a Corollary below.

Corollary 4.13. *The functions on $\mathbb{A}^{n|k,m}$ are isomorphic as a ring to*

$$\left\{ \sum_{I,J,T \geq 0} a_{I,J,T} x^I \xi^J y^T / T! \right\}$$

where x_i and y_i are even and ξ_i are odd, and $a_{I,J,T} \in \mathbb{Z}_p$ with $a_{I,J,T} \to 0$ as $I \to \infty$.

Proof. Using Theorems 4.10 and 4.12 with induction we see that the ring of functions on $\mathbb{A}^{0|k,0}$ is $\mathbb{Z}_p \langle y_1, ..., y_k \rangle$. Gluing this fact with the proven part of Theorem 3.8 using Theorem 4.12 we obtain the desired result. $\square$

Denoting the functions described in the Theorem 4.12 by $R\langle x \rangle$ and observing that the functions on X_U are given by R we are done by the following Lemma.

Lemma 4.14. *The natural map $\pi : \Omega_{R\langle x \rangle} \to \Omega_R$ is a quasi-isomorphism.*

Proof. In fact we show that the equally natural map $\rho : \Omega_R \to \Omega_{R\langle x \rangle}$ is a homotopy inverse. Note that $\pi \circ \rho = \mathrm{Id}_{\Omega_R}$, let $F = \rho \circ \pi$, we must show that there is a homotopy h such that $\mathrm{Id}_{\Omega_{R\langle x \rangle}} - F = d \circ h + h \circ d$.

It follows immediately from the abstract definitions and by using Theorem 4.12, that any $w \in \Omega^s_{R\langle x \rangle}$ can be written uniquely as

$$w = \sum_{i=0}^{\infty} \alpha_i x^i / i! + \sum_{i=0}^{\infty} \beta_i x^i / i! dx$$

where $\alpha_i \in \Omega^s_R$ and $\beta_i \in \Omega^{s-1}_R$. Let

$$h(w) = (-1)^{s-1} \sum_{i=0}^{\infty} \beta_i x^{i+1} / (i+1)!,$$

then a straightforward calculation shows that h is the desired homotopy. $\square$

At this point we have proven the main result of the section, namely:

Theorem 4.15. *Let V be a smooth projective variety over $\mathbb{Z}_p$, let X_V be the associated p-adic superspace, and $\widetilde{X_V}$ the DP neighborhood of V inside $\mathbb{P}^n$, then*

$$H_{dR}(V) \simeq DR_{\mathbb{Z}_p}(\widetilde{X_V}).$$

Corollary 4.16. *Let V be a smooth projective variety over $\mathbb{Z}_p$, then one has an action of Frobenius on $H_{dR}(V)$.*

Proof. By Lemma 3.13 we have an action of Frobenius on $\widetilde{X_V}$ and so on its de Rham cohomology, which is isomorphic to $H_{dR}(V)$. $\square$

Corollary 4.17. *Let W be a smooth projective variety over $\mathbb{F}_p$, then the crystalline cohomology of W is isomorphic to the de Rham cohomology of the p-adic superspace $\widetilde{W}$ (the Grassmann neighborhood of W in $\mathbb{P}^n_{\mathbb{Z}_p}$), i.e.,*

$$H_{crys}(W) \simeq DR_{\mathbb{Z}_p}(\widetilde{W}).$$

Proof. Let V be any smooth projective lifting of W to $\mathbb{Z}_p$, i.e. $W = V|_{\mathbb{F}_p}$. Then by Theorem 4.15, $H_{dR}(V) \simeq DR_{\mathbb{Z}_p}(\widetilde{X_V})$. However $\widetilde{X_V} = \widetilde{W}$, and $H_{crys}(W) \simeq H_{dR}(V)$, so we are done. $\square$

It is worth noting that the homotopy inverse ρ that was used in the proof of Lemma 4.14 can not be realized (in general) as a restriction of a global map $r^* : DR_{\mathbb{Z}_p}(X_V) \to DR_{\mathbb{Z}_p}(\widetilde{X_V})$.[21] Geometrically speaking we may not in general have a global retraction r of $\widetilde{X_V}$ onto X_V, its existence would ensure that $i^* : DR_{\mathbb{Z}_p}(\widetilde{X_V}) \to DR_{\mathbb{Z}_p}(X_V)$ is an isomorphism of filtered modules with respect to the Hodge filtration. Consequently, the canonical lift of the Frobenius morphism Fr to $DR_{\mathbb{Z}_p}(X_V)$ would preserve the Hodge filtration $F^\bullet DR_{\mathbb{Z}_p}(X_V)$. Furthermore, consider the following local computation. Let x be a local function on $\widetilde{X_V}$, then $Fr(x) = x^p + py$, where y is some other local function, so that

$$Fr : f dx_1 ... dx_s \mapsto p^s Fr(f)(x_1^{p-1} dx_1 + dy_1)...(x_s^{p-1} dx_s + dy_s),$$

i.e. under the assumption that a global retraction exists

$$Fr(F^s DR_{\mathbb{Z}_p}(X_V)) \subset p^s F^s DR_{\mathbb{Z}_p}(X_V).$$

Neither the invariance of the Hodge filtration under Fr nor the p-divisibility estimate need hold in the absence of the global retraction, in Sec. 5 we obtain some weaker p-divisibility estimates that hold in general.

[21] In contrast with π which is a restriction of a global map $i^* : DR_{\mathbb{Z}_p}(\widetilde{X_V}) \to DR_{\mathbb{Z}_p}(X_V)$.

5 The Frobenius map and the Hodge filtration.

In this section we essentially follow B. Mazur[11] with some differences in the point of view (that is we find it more conceptual to think of DP ideals and their DP powers). We begin by recalling a definition.

Definition 5.1. Let I in A be a DP ideal, then the nth DP power of I, denoted $I^{[n]}$, is the ideal generated by the products $x_1^{n_1}/n_1!...x_k^{n_k}/n_k!$ with $x_i \in I$ and $\sum n_i \geq n$.

We point out that the Hodge filtration on the de Rham cohomology of X is simply the filtration on the functions on ΠTX given by the DP-powers of the DP ideal I_X of X in ΠTX. More precisely:

Definition 5.2. For a p-adic superspace X, define a filtration, $F_H^\bullet$ on $\Omega_{X/\mathbb{Z}_p}$ by setting $F_H^i\Omega_{X/\mathbb{Z}_p} = I_X^{[i]}$. This filtration descends to $DR_{\mathbb{Z}_p}(X)$[22] and let us still denote it by $F_H^\bullet$ there.

The following Lemma is immediate.

Lemma 5.3. *Let V be a smooth projective variety over $\mathbb{Z}_p$, then the isomorphism*

$$H_{dR}(V) \simeq DR_{\mathbb{Z}_p}(X_V)$$

is compatible with the Hodge filtration on the left and $F_H^\bullet$ on the right.

Remark. Because of the above Lemma we will use the notation $F_H^\bullet$ to denote also the Hodge filtration on $H_{dR}(V)$.

However the Frobenius map that we are interested in is defined on $DR_{\mathbb{Z}_p}(\widetilde{X_V})$ not $DR_{\mathbb{Z}_p}(X_V)$. And while the two are isomorphic as shown previously, this isomorphism is not compatible with $F_H^\bullet$. To fix this problem we proceed as follows: replace the $F_H^\bullet$ filtration on $DR_{\mathbb{Z}_p}(\widetilde{X_V})$ which is given by the ideal of $\widetilde{X}$ in $\Pi T\widetilde{X}$ with the one given by the DP-powers of the ideal of X itself in $\Pi T\widetilde{X}$.

Definition 5.4. Let $X \subset \widetilde{X} \subset \Pi T\widetilde{X}$ be as above, define a filtration $F_{DP}^\bullet$ on $DR_{\mathbb{Z}_p}(\widetilde{X})$ as the induced filtration from $\Omega_{\widetilde{X}/\mathbb{Z}_p}$, where

$$F_{DP}^i\Omega_{\widetilde{X}/\mathbb{Z}_p} = I_X^{[i]}.$$

[22] The action of $\mathbb{A}^{0,1}$ on ΠTX preserves X, thus the differential preserves I_X and its DP powers.

In the particular case, namely the setting of Lemma 4.14 (that is the key step in proving the general case), the definition above becomes: $F^{\bullet}_{DP}\Omega_{R\langle x\rangle}$ is defined by $w \in F^s\Omega_{R\langle x\rangle}$ if

$$w = \sum_{i=0}^{\infty} \alpha_i x^i/i! + \sum_{i=0}^{\infty} \beta_i x^i/i! dx$$

where $\alpha_i \in \Omega_R^{\geqslant s-i}$ and $\beta_i \in \Omega_R^{\geqslant s-1-i}$. It is then not hard to show (using the observation that the homotopy of Lemma 4.14 preserves the new filtration) that $i^* : DR_{\mathbb{Z}_p}(\widetilde{X_V}) \to DR_{\mathbb{Z}_p}(X_V)$ is an isomorphism of filtered modules where $DR_{\mathbb{Z}_p}(\widetilde{X_V})$ is endowed with the new filtration $F^{\bullet}_{DP}$ and $DR_{\mathbb{Z}_p}(X_V)$ has the old filtration $F^{\bullet}_H$. Thus we have a refinement of Theorem 4.15.

Theorem 5.5. *Let V be a smooth projective variety over $\mathbb{Z}_p$, let X_V be the associated p-adic superspace, and $\widetilde{X_V}$ the DP neighborhood of V inside $\mathbb{P}^n$, then*

$$H_{dR}(V) \simeq DR_{\mathbb{Z}_p}(\widetilde{X_V})$$

is an isomorphism of filtered modules with the Hodge filtration on the left and $F^{\bullet}_{DP}$ on the right.

Though Fr does not preserve it, $F^{\bullet}_{DP}$ is useful in the computation of divisibility estimates. Notice that $Fr(I_X^{[k]}) \subset p^{[k]}\mathcal{O}_{\widetilde{X}}$ [23], where $I_X^{[k]}$ is the k-th DP power of the ideal (also denoted by I_X) of X in $\widetilde{X}$ and $[k] = \min_{n\geqslant k}\mathrm{ord}_p(\frac{p^n}{n!})$ (thus $(p)^{[k]} = (p^{[k]})$). Recalling the discussion at the end of Sec. 4.3, we see that

$$Fr(F^s_{DP} DR_{\mathbb{Z}_p}(\widetilde{X_V})) \subset p^{[s]} DR_{\mathbb{Z}_p}(\widetilde{X_V}).$$

In particular if $p > \dim(V)$ then the square brackets can be removed from s and we obtain the following Lemma.

Lemma 5.6. *Let V be a smooth projective variety over $\mathbb{Z}_p$ such that $p > dim(V)$, then*

$$Fr(F^s_H H_{dR}(V)) \subset p^s H_{dR}(V).$$

[23] A very useful formula to keep in mind is $\mathrm{ord}_p n! = \sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right] = \frac{n-S(n)}{p-1}$, where $S(n)$ is the sum of digits in the p-adic expansion of n.

A slightly finer statement can be derived from the above observations, one actually has

$$Fr(F^s_{DP}DR_{\mathbb{Z}_p}(\widetilde{X_V})) \subset \sum_{j<s} p^{[s-j]}Fr(F^j_{DP}DR_{\mathbb{Z}_p}(\widetilde{X_V})) + p^s DR_{\mathbb{Z}_p}(\widetilde{X_V})$$
$$\subset pFr(DR_{\mathbb{Z}_p}(\widetilde{X_V})) + p^s DR_{\mathbb{Z}_p}(\widetilde{X_V}).$$

The latter was sufficient for Mazur to establish a conjecture of Katz.

By analogous reasoning one can introduce new filtrations on the cohomologies of X and $\widetilde{X}$ by considering the DP ideal of $X|_{\mathbb{F}_p}$ in ΠTX and the DP ideal of $X|_{\mathbb{F}_p}$ in $\Pi T\widetilde{X}$. The canonical isomorphism is now an isomorphism of filtered modules with respect to these new filtrations and they are preserved by Fr. The new filtration on $DR_{\mathbb{Z}_p}(X_V)$ contains the Hodge filtration and satisfies the same divisibility conditions. In fact it can be easily described as follows (let us work with $H_{dR}(V)$ since it is the same as $DR_{\mathbb{Z}_p}(X_V)$). Let $F^\bullet_H$ denote the usual Hodge filtration on $H_{dR}(V)$, then the new filtration $F^\bullet_N$ can be described thus: $F^n_N H_{dR}(V) = \sum_{s+t\geqslant n} p^{[s]} F^t_H H_{dR}(V)$.

6 Appendix.

We investigate a property of a p-adic superspace that we call prorepresentability. It allows us to pass from particular examples that we considered in this paper (namely p-adic superspaces that arise in dealing with usual varieties over $\mathbb{Z}_p$) to a more general class of p-adic superspaces that nevertheless share a lot of properties with our examples.

Recall that we have defined a map of p-adic superspaces as a map of the defining functors. It is well known (Yoneda Lemma) that the set $\mathrm{Hom}(F,G)$ of natural transformations from a functor F to a functor G can be easily calculated[24] if F is representable, i.e. F is isomorphic to the functor h_A, where

$$h_A(X) = \mathrm{Hom}(A, X).$$

Namely, in this case we have

$$\mathrm{Hom}(F,G) = G(A).$$

[24] The source category does not matter and the target category is *Sets*.

Definition 6.1. We say that a functor F is prorepresentable if it is isomorphic to a colimit of representable functors:

$$F = \varinjlim_{A \in D} h_A$$

for some diagram D.

Thus

$$\mathrm{Hom}(F, G) = \varprojlim_{A \in D} G(A),$$

i.e. is a limit of the sets $G(A)$, by Yoneda Lemma and continuity of $\mathrm{Hom}(-,-)$. Notice that in the above definitions we use a general definition of limits and colimits, a concise reference is [10]. It is important to emphasize that the definition of a prorepresentable functor in [13] is much more restrictive.

Definition 6.2. We say that a p-adic superspace X is prorepresentable, if its defining functor is locally prorepresentable, i.e. $[X]$ has a cover by open U_i such that the defining functors of $X|_{U_i}$ are prorepresentable.

One can show that the property of a p-adic superspace X being prorepresentable is preserved by passing to the odd tangent space ΠTX.

Theorem 6.3. *If a p-adic superspace X is prorepresentable then so is ΠTX.*

Proof. This statement is local, so we may assume that X is prorepresentable as a functor. Denote by F_ξ the endo-functor of Λ that takes an object $A \in \Lambda$ to $A[\xi]$, i.e. adjoins an odd variable. Note that $\Pi TX = X \circ F_\xi$. Observe that F_ξ extends to *Super*. *Super* is closed under limits and F_ξ commutes with limits since it has a left adjoint $\Omega^\bullet_-$. Thus we may assume that X is representable and since $\Omega^\bullet_-$ descends to Λ we are done. $\square$

One can show that all the p-adic superspaces we consider in this paper are prorepresentable in our sense. Here we give a detailed proof of this fact for the most important functor $\widetilde{pt}$, which is indeed prorepresentable (not just locally prorepresentable). In a similar way one can prove the local prorepresentability of functors corresponding to other superspaces considered in the present paper. As an application we show how to describe the functions on $\widetilde{pt}$ (Theorem 4.10) using the above ideas.

Consider the ring $C[\xi_i]_{i=1}^{2n}$, i.e. a supercommutative ring obtained from a commutative ring C by adjoining $2n$ odd anticommuting variables ξ_j. Let G be the group acting on $C[\xi_i]_{i=1}^{2n}$ generated by

$$\xi_{2k-1} \mapsto \xi_{2k},\ \xi_{2k} \mapsto -\xi_{2k-1}$$

and

$$\xi_{2k-1}, \xi_{2k} \mapsto \xi_{2k'-1}, \xi_{2k'}$$

$$\xi_{2k'-1}, \xi_{2k'} \mapsto \xi_{2k-1}, \xi_{2k}.$$

Lemma 6.4. *The subring of G-invariants in $C[\xi_i]_{i=1}^{2n}$ is spanned by $w_n^k/k!$ for $k = 0, ..., n$ where $w_n = \xi_1\xi_2 + ... + \xi_{2n-1}\xi_{2n}$. That is*

$$(C[\xi_i]_{i=1}^{2n})^G = C\langle y\rangle/(y)^{n+1}.$$

Proof. Proceed by induction on n. For $n = 0$ there is nothing to prove. Let it be true for n. Note that

$$\begin{aligned} &C[\xi_i]_{i=1}^{2(n+1)} \\ &\quad = C[\xi_i]_{i=1}^{2n} \oplus C[\xi_i]_{i=1}^{2n}\xi_{2n+1} \oplus C[\xi_i]_{i=1}^{2n}\xi_{2n+2} \oplus C[\xi_i]_{i=1}^{2n}\xi_{2n+1}\xi_{2n+2}. \end{aligned}$$

Let $x \in C[\xi_i]_{i=1}^{2(n+1)}$ be G-invariant, then the element of G that "switches" ξ_{2n+1} and ξ_{2n+2} ensures that

$$x \in C[\xi_i]_{i=1}^{2n} \oplus C[\xi_i]_{i=1}^{2n}\xi_{2n+1}\xi_{2n+2}.$$

Considering the part of G that acts on $C[\xi_i]_{i=1}^{2n}$ only and using the induction hypothesis we see that

$$x = \sum a_k w_n^k/k! + b_{k-1}w_n^{k-1}/(k-1)!\xi_{2n+1}\xi_{2n+2}.$$

Since the action of G is degree preserving each homogeneous component of x is also G invariant, thus

$$a_k w_n^k/k! + b_{k-1}w_n^{k-1}/(k-1)!\xi_{2n+1}\xi_{2n+2}$$

is G-invariant.

The element of G that switches ξ_1, ξ_2 and ξ_{2n+1}, ξ_{2n+2} ensures that

$$a_k = b_{k-1},$$

so that

$$x = \sum a_k w_{n+1}^k/k!.$$

□

Observe that if a group G acts on a set X, then the fixed points subset $X^G \subset X$ can be represented as a limit of the diagram in $Sets$ consisting of two copies of X and the arrows given by the elements of G. This is where Lemma 6.4 is used in Theorem 6.5 below.

Theorem 6.5. *The functor defining the p-adic superspace $\widetilde{pt}$ is prorepresentable.*

Proof. Consider the diagram in Λ consisting of objects

$$\{(\mathbb{Z}_p/p^n\mathbb{Z}_p)[\xi_i]_{i=1}^{2m} | m, n \geq 0\} \coprod \{(\mathbb{Z}_p/p^n\mathbb{Z}_p)[\xi_i]_{i=1}^{2m} | m, n \geq 0\}$$

with morphisms between the copies given by elements of G, and the rest of the morphisms given by the usual projections

$$(\mathbb{Z}_p/p^n\mathbb{Z}_p)[\xi_i]_{i=1}^{2m} \to (\mathbb{Z}_p/p^{n'}\mathbb{Z}_p)[\xi_i]_{i=1}^{2m}$$

for $n > n'$ and

$$(\mathbb{Z}_p/p^n\mathbb{Z}_p)[\xi_i]_{i=1}^{2m} \to (\mathbb{Z}_p/p^n\mathbb{Z}_p)[\xi_i]_{i=1}^{2m'}$$

for $m > m'$ mapping the extra $\{\xi_j\}_{j>2m'}$ to 0.

It follows from Lemma 6.4 that while the limit of the above diagram does not exist in Λ, it exists in the category $Super$, where Λ is a full subcategory, and it is equal to the ring with divided powers $\mathbb{Z}_p \langle x \rangle$. Correspondingly, after passing to the category of functors from Λ to $Sets$, the functor

$$\widetilde{pt} = \mathrm{Hom}_{Super}(\mathbb{Z}_p \langle x \rangle, -)$$

is seen to be the colimit of representable functors. □

Remark. To summarize the above proof, the main ingredient is the observation that the functor $\widetilde{pt}$ extends to $Super$ where it is representable. Furthermore, the representing object $\mathbb{Z}_p \langle x \rangle$ is a limit of a diagram of objects in Λ. This representing diagram is not unique, however it does not prevent us from easily proving Theorem 4.10, i.e. computing the functions on $\widetilde{pt}$ below.

Corollary 6.6. *The functions on the p-adic superspace $\widetilde{pt}$ are $\mathbb{Z}_p \langle x \rangle$.*

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